Consensus of High-Order Multi-Agent Systems with Binary-Valued Communications and Switching Topologies

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***Abstract*—This paper studies the consensus problem of high-order multi-agent systems (MASs) with binary-valued communications and switching topologies. In order to overcome the challenge of unknown states caused by binary-valued communications, this paper constructs an estimation-based consensus algorithm. First, a recursive projection identification algorithm is presented to estimate the neighbors' states dynamically. Then, based on these estimates, a consensus law is designed. By constructing and analyzing two Lyapunov functions about estimation error and state error, this paper establishes their relation, to overcome the difficulty resulting from the coupling of the estimation and control and less information due to switching topologies. Under the condition of jointly connected topologies, it is proven that by properly selecting the step coefficient, the estimates of states can converge to the true states with a convergence rate as the reciprocal of the iteration times. Besides, the MAS is proved to achieve weak consensus and the consensus rate is also established as the reciprocal of the iteration times. Finally, a simulation example is given to validate the algorithm.**

***Index Terms*—multi-agent system, high-order, binary-valued communication, switching topology, consensus, recursive projection identification algorithm**

I. Introduction

In recent years, the consensus problem of multi-agent systems (MASs) has attracted increasing attention from scholars across various fields [1]–[8], such as swarm formation for unmanned aerial vehicles in engineering fields [1], the reputation consensus of mobile nodes in communication fields [8], and so on. In swarm scenarios, all agents dynamically adjust their positions and orientations relative to neighboring agents to ensure a common heading direction.

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At the start, the consensus problems are investigated with accurate communications and fixed topologies, such as [9]–[11]. Furthermore, [12] proposed the necessary and sufficient conditions of average-consensus in the noise-free case and asymptotic unbiased mean square average-consensus in the case of stochastic noises.

However, due to the limited capacity of the communication channel, only limited data can be transmitted over the channel per unit of time. Therefore, in each time interval, only limited bits of data can be exchanged between agents, which is also called quantized information [13]–[15]. Since the wide application of digital networks, the consensus problem over capacity-limited networks has attracted a lot of interest. For example, [16]–[20] considered the consensus problem with quantized communication, which only needs finite bits in transmission.

Furthermore, binary-valued information is a specialized form of quantized information, with the transmission of just one bit by simplifying communication into only true or false states. Binary-valued communication significantly cuts more communication costs than others, contributing to its widespread and efficient application. As a result, some works on the consensus problem have appeared based on binary-valued communications [21]–[24]. [22] constructed a two-time-scale control algorithm and proved that the MAS can achieve weak consensus and mean square consensus. [23] proposed a consensus algorithm based on recursive projection and gave the mean-square consensus rate. [24] expanded the system of [23] to high-order MASs, but with an orthogonal limitation on the coefficient matrices.

It is worth noticing that all the consensus works mentioned above are for the fixed topology case. Actually, the topologies of multi-agent networks usually switch over time in practical networks due to the interference of some external elements and the changes in the current circumstances. There are some works that investigate the consensus problem of MASs with switching topologies as well. For example, [25] studied the case with accurate communications and switching topologies that are jointly connected, simulating a simple multi-agent collaboration model. [26] employed a recursive projection identification algorithm to develop a control law and proved that the first-order switching MAS with binary-valued communications can reach consensus with this control law. [27] proposed a control law based on an adaptive encoding-decoding scheme, demonstrating that the high-order switching MAS without communication noises can exponentially achieve consensus with finite bits of information.

Moreover, high-order systems play a critical role in various practical applications, such as formation control [1]–[2], social networks [28], and so on. On the other hand, given the widespread use of digital communication, binary-valued communication holds significant practical value due to its ability to reduce communication costs compared with other methods substantially. However, binary-valued communications and switching topologies result in less transmitted information, making theoretical analysis more complex, while the inclusion of high-order systems leads to greater complexity in the dynamics of MASs. Consequently, it is imperative and challenging to address the consensus problem of high-order MASs under binary-valued communications and switching topologies, which is precisely the purpose of this paper. The main contributions of this paper are as follows:

- This paper is the first to address the consensus problem of MAS with a high-order system, binary-valued communication, and switching topology simultaneously. In contrast to the existing works [24] and [26], this paper has a more general model. To be specific, the states of agents in high-order MASs are dynamic even if the control input is absent, whereas the states of first-order MASs in [26] are static. Therefore, each agent needs to estimate its neighbors' states dynamically in this paper, which makes state estimation more complicated. Besides, by jointly analyzing the structure feature of the topology graph and system model, this paper relaxes the limitation on coefficient matrices in [24] and only requires the system to be marginally stable. On the other hand, this paper only requires binary-valued transmission and has a lower communication cost than [27].
- An estimation-based consensus algorithm, consisting of estimation and control, is designed. First, to overcome the challenge of unknown states caused by binary-valued communications, a recursive projection identification algorithm is presented to estimate the neighbors' states. Then, a consensus control law is designed based on the estimates of neighbors' states. It is worth mentioning that this paper introduces an adjustable coefficient into the controller that removes the connectivity limitation on the graph structure in [26].
- Two Lyapunov functions are designed to analyze the consensus of all agents and the convergence of the estimates, respectively. Through the analysis of these two Lyapunov functions, this paper establishes the relation between them to overcome the difficulty resulting from the coupling of the estimation and control. At the proper step coefficient, it is proven that the estimation errors of neighbors' states can converge to zero and the MAS can achieve weak consensus. Furthermore, even without connectivity constraint on the graph structure as mentioned in [26], the convergence rate of the estimation errors and the consensus rate can still reach the reciprocal of the iteration number as [26].

The remainder of this paper is organized as follows: Section II gives the preliminaries of basic concepts and graph theory and describes the consensus problem. Section III introduces the estimation-based consensus algorithm. The main results of this paper are presented in Section IV, which include the main convergence and consensus results. In Section V, a simulation example is given. Section VI is the summary and prospect of this paper.

II. Preliminaries and Problem Formulation

In this section, we first give some basic concepts in matrix and graph theory, and subsequently formulate the system model and the consensus problems investigated in this paper.

A. Basic concepts

We use $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times m}$ to denote n-dimensional column vector and $n\times m$-dimensional real matrix, respectively. Denote $\vec{0}_m = [0,\dots,0]^T \in \mathbb{R}^m$, where the notation T denotes the transpose operator. Moreover, we denote $\|x\| = \|x\|_2$ and $\|A\| = \sqrt{\lambda_{\max}(AA^T)}$ as the Euclidean norm of vector and matrix, respectively, where $\lambda_{\max}(\cdot)$ denotes the largest eigenvalue of the matrix. Correspondingly, $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of the matrix. For symmetric matrices $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times m}$, $A \geq B$ represents that $A - B$ is a positive semi-definite matrix. diag$\{\cdot\}$ denotes the block-diagonal matrix. And, for arbitrary matrices $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product of A, B is defined as

$$A \otimes B \triangleq \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Besides, the mathematical expectation is denoted as $E[\cdot]$.

B. Graph theory

In order to describe the relation between agents, we introduce a time-varying topology $G_{m(t)} = (N_0, E_{m(t)})$, where $m(t) \in \{1,2,\dots,h\}$ is a time-varying function, $N_0 = \{1,\dots,N\}$ is the set of agents, and $E_{m(t)} \subseteq N_0 \times N_0$ is the ordered edges set of the topology $G_{m(t)}$. Moreover, assume $G_{m(t)} \in \{G_1, G_2, \dots, G_h\}$ and $E_{m(t)} \in \{E_1, E_2, \dots, E_h\}$. Denote $N_i^{m(t)}$ as the neighbor set of the agent i in the topology $G_{m(t)}$. Denote the adjacency matrix of the N agents at time t as $A_{m(t)}$, where each element of the matrix $A_{m(t)}$ satisfies $a_{ij}^{m(t)} = 1$ if $(i,j) \in E_{m(t)}$, else $a_{ij}^{m(t)} = 0$. Denote the degree matrix of the N agents at time t as $D_{m(t)}$, where $D_{m(t)} = \text{diag}\{d_1^{m(t)}, d_2^{m(t)}, \dots, d_N^{m(t)}\}$ and $d_i^{m(t)}$ is the degree of agent i at time t. Then, the Laplace matrix of $G_{m(t)}$ is defined as $L_{m(t)} = D_{m(t)} - A_{m(t)}$.

C. Problem formulation

Consider the following MAS with N agents at time t:

$$x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1,\dots,N, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are constant matrices, $x_i(t) \in \mathbb{R}^n$ is the state of the agent i at time t, and $u_i(t) \in \mathbb{R}^m$ is the control input of the agent i at time t.

Remark 1: As mentioned earlier, the consensus problem of high-order MASs has wide applications. Such as formation control issues on a plane or in space, where agent states $x_i(t)$ are typically represented as two-dimensional or three-dimensional vectors [1]–[2]. Besides, high-order system models are frequently encountered in social networks [28], such as when people simultaneously participate in discussions on multiple topics with state $x_i(t)$ in (1). Consequently, compared with first-order systems, high-order systems are more general and more commonly used in the real world. In addition, the system model of the form (1) is a typical high-order MAS, widely employed in [24], [27], [29] and [30].

The agent i receives the following binary-valued information with communication noise from its neighbor j:

$$\begin{cases} y_{ij}(t) = x_j(t) + d_{ij}(t), \\ s_{ij}(t) = \mathbb{1}_{\{y_{ij}(t)\le c_{ij}\}}, \end{cases} \tag{2}$$

where the agent j is the neighbor of the agent i at time t, $d_{ij}(t) \in \mathbb{R}^n$ is the communicating noise, $y_{ij}(t) \in \mathbb{R}^n$ is the virtual output, $c_{ij} \in \mathbb{R}^n$ is the threshold value, $s_{ij}(t)$ is the binary-valued information that the agent i collects from its neighbor j, $\mathbb{1}_{\{a\le c\}}$ is the indicative function defined as:

$$\mathbb{1}_{\{a\le c\}} = \left[\mathbb{1}_{\{a(1)\le c(1)\}}, \mathbb{1}_{\{a(2)\le c(2)\}}, \ldots, \mathbb{1}_{\{a(n)\le c(n)\}}\right]^T,$$

with $a = [a(1), a(2), \ldots, a(n)]^T$, $c = [c(1), c(2), \ldots, c(n)]^T$, and for $k = 1, 2, \ldots, n$,

$$\mathbb{1}_{\{a(k)\le c(k)\}} = \begin{cases} 1, & a(k) \le c(k), \\ 0, & a(k) > c(k). \end{cases}$$

Remark 2: The communication form of $\mathbb{1}_{\{a\le c\}}$ is commonly used in the communication field, such as [31]–[34]. In order to provide a clear understanding of the definition of $\mathbb{1}_{\{a\le c\}}$, an example is given as follows: If $a = [-1, 2, 5, -3, 0]^T$, $c = [0, 0, 0, 0, 0]^T$, then, $\mathbb{1}_{\{a\le c\}} = [1, 0, 0, 1, 1]^T$.

In order to proceed with our analysis, we introduce some assumptions about the graph, the noise, and the system coefficients.

Assumption 1: $\{G_1, G_2, \ldots, G_h\}$ are jointly connected and G_i emerges at time t with a probability $p_i(> 0)$, for $i = 1, 2, \ldots, h$, where $\sum_{i=1}^h p_i = 1$.

Assumption 2: The noise $d_{ij}(t)$ is independent identically normally distributed as $N(0, \delta^2 I_n)$ for i, j, t, which implies that each element of $d_{ij}(t)$ has the same distribution function $F(\cdot)$ and the associated density function $f(\cdot)$, respectively.

Assumption 3: $L_{m(t)}$ and $d_{ij}(t)$ are independent. Besides, $L_{m(t)}$ and $L_{m(l)}$ are independent for $t \ne l$.

Assumption 4: The system matrix A is an orthogonal matrix, and B is of full row rank.

Remark 3: Actually, by Remark 2.3 of [35], we know that Assumption 4 can be relaxed to the case where the matrix A is neutrally stable, which is common in the model assumptions and practical applications, such as [27], [35]–[38]. If the matrix A is neutrally stable but not orthogonal, there is a nonsingular matrix O such that $\tilde{A} = O^{-1}AO$ is orthogonal. Let $\tilde{x}_i(t) = O^{-1}x_i(t)$ and $\tilde{B} = O^{-1}B$. Then, $\tilde{x}_i(t+1) = \tilde{A}\tilde{x}_i(t) + \tilde{B}u_i(t)$, where $\tilde{A}$ is orthogonal, $\tilde{B}$ is of full row rank.

Moreover, Assumption 4 can be relaxed to the case that the matrix A is marginally stable. In detail, by Remark 2.2 of [35], we know that if the matrix A is marginally stable but not neutrally stable, then there is a nonsingular matrix T such that $TAT^{-1} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$, where A_s is stable and A_u is neutrally stable. And as [39] says, since the MAS with a stable coefficient matrix A_s can reach consensus even if the control input is zero, we just need to focus on the neutrally stable part A_u. In contrast to Assumption 2 in [24], Assumption 4 in this paper is more general, which relaxes the restrictions on the coefficient matrices.

Now, we introduce the concept of weak consensus and the problem to be studied.

Definition 1: ([40, Definition 2] **Weak Consensus**). Denote $x_i(t)$ as the state of agent i at time t, where $i = 1, \ldots, N$. For all agents, if $x_i(t)$, $i = 1, \ldots, N$, satisfy:

(1) $E\|x_i(t)\|^2 < \infty$, $i = 1, \ldots, N$; and

(2) $\lim_{t\to\infty} E\|x_i(t) - x_j(t)\|^2 = 0$, $i, j \in \{1, \ldots, N\}$.

Then, the agents are said to achieve weak consensus.

Problem: The goal of this paper is to design a controller $u_i(t)$ based on binary-valued communications $s_{ij}(t)$ and switching topologies $G_{m(t)}$ to achieve weak consensus.

III. Algorithm design

This section focuses on the design of a consensus control algorithm. In general, the consensus control is designed by using the accurate states of the neighbors, as mentioned in [9]–[12] and [25]. However, in this paper, the agent can only obtain binary-valued communications from its neighbors. A straightforward idea is to replace the accurate states of neighbors with their estimates, so each agent should estimate its neighbors' states firstly by the binary-valued communications, and then, design the consensus control based on these estimates.

Based on the above idea, we propose an estimation-based consensus algorithm involving both estimation and control, named as Algorithm 1.

Algorithm 1 Estimation-Based Consensus Algorithm

i) **Initiation:** Denote the integer $t_0(> 0)$ as the initial time. $x_i(t_0+1) = x_i^0$ is the initial state of the agent i, $\hat{x}_{ij}(t_0) = x_{ij}^0$ is the initial estimate of the agent j estimated by the agent i. And, denote M as the upper boundary for the norm of these initial values, i.e., $M \ge \|x_i^0\|$, $M \ge \|x_{ij}^0\|$. For $t \ge t_0 + 1$, the algorithm is as follows.

ii) **Observation:** each agent i gets the binary-valued observations from its neighbors

$$\begin{cases} y_{ij}(t) = x_j(t) + d_{ij}(t), \\ s_{ij}(t) = \mathbb{1}_{\{y_{ij}(t)\le c_{ij}\}}, \end{cases}$$

where $j \in N_i^{m(t)}$, $i = 1, \ldots, N$, $m(t) \in \{1, \ldots, h\}$.

iii) **Estimation:** each agent i estimates the state of its neighbor agent j at time t by

$$\hat{x}_{ij}(t) = \Pi_M\Big\{A\hat{x}_{ij}(t-1)+\frac{\beta}{t}\Big(\mathcal{F}\big(c_{ij}-A\hat{x}_{ij}(t-1)\big)-s_{ij}(t)\Big)\Big\}, \tag{3}$$

where $j \in N_i^{m(t)}$, β is the step coefficient for estimation updating, $\mathcal{F}(z) = [F(z_1), \ldots, F(z_n)]^T$ for any $z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{R}^n$, $\Pi_M(\cdot)$ is a projection mapping defined as

$$\Pi_M(\zeta) = \arg\min_{\|\xi\|\le M} \|\zeta - \xi\|, \forall \zeta \in \mathbb{R}^n. \tag{4}$$

iv) **Controller:** based on these estimates, each agent i designs its control by

$$u_i(t) = \frac{\gamma}{(t+1)d_{\max}} B^T A \sum_{j\in N_i^{m(t)}} \big(\hat{x}_{ij}(t) - x_i(t)\big), \tag{5}$$

where $d_{\max} = \max_{1\le i\le N, 1\le m(t)\le h} \{d_i^{m(t)}\}$, $0 < \gamma < \infty$.

v) **Repeat:** Let $t = t + 1$, go back to Step ii).

Remark 4: At the first step of Algorithm 1, the initial value of the estimate can be chosen arbitrarily, i.e., can be any given real number. The boundary M is then selected according to the initial values of states and estimates, which is a piece of global information of the MAS. By using the projection operator with boundary M, both the estimates and values of agents' states are constrained within the bounds of M as outlined in Algorithm 1. In other words, for any given system, the estimation-based consensus algorithm designed here can make the system consensus in the range determined by the initial values.

Remark 5: The projection mapping Π_M is used to guarantee the boundness of the estimates and good convergence effect in the initial iterative process of the algorithm, which is common in binary-valued identification, such as [23], [26], and [41]. It is used to construct the damping compression coefficients in the convergence analysis of the algorithm designed by the noise distribution function under binary-valued data.

Besides, as [41, Proposition 6] says, the projection mapping given by (4) has the following property:

$$\|\Pi_M(x_1) - \Pi_M(x_2)\| \le \|x_1 - x_2\|, \forall x_1, x_2 \in \mathbb{R}^n.$$

Remark 6: By (1) and (5), the state of the agent i is updated as $x_i(t+1) = Ax_i(t) + \frac{\gamma BB^T A}{(t+1)d_{\max}} \sum_{j\in N_i^{m(t)}} \big(\hat{x}_{ij}(t) - x_i(t)\big)$.

For the convenience of the subsequent analysis, we rewrite the above estimation and update in vector form.

Firstly, define $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T \in \mathbb{R}^{nN}$.

Then, denote the jointly connected topology formed by $G_1, G_2, \ldots, G_h$ as $G = (N_0, E)$, where $E = E_1 \cup \cdots \cup E_h$ is the set of all the edges. Next, we consider the agent i in the jointly connected graph G, denote d_i as its degree and N_i as the set of its neighbors and $d = \sum_{i=1}^N d_i$. Based on these, denote

$$\begin{aligned}\hat{x}(t) =&[\hat{x}_{1r_1}^T(t), \hat{x}_{1r_2}^T(t), \ldots, \hat{x}_{1r_{d_1}}^T(t), \ldots, \hat{x}_{ir_{d_1+\ldots+d_{i-1}+1}}^T(t),\\ &\ldots, \hat{x}_{ir_{d_1+\ldots+d_i}}^T(t), \ldots, \hat{x}_{Nr_{d_1+\ldots+d_N}}^T(t)]^T \in \mathbb{R}^{nd},\end{aligned}$$

where $r_{d_1+d_2+\ldots+d_{i-1}+1}, \ldots, r_{d_1+\ldots+d_i} \in N_i$ for $i = 1, 2, \ldots, N$.

Similarly, denote

$$\begin{aligned}S(t) =&[s_{1r_1}^T(t), s_{1r_2}^T(t), \ldots, s_{1r_{d_1}}^T(t), \ldots, s_{ir_{d_1+\ldots+d_{i-1}+1}}^T(t),\\ &\ldots, s_{ir_{d_1+\ldots+d_i}}^T(t), \ldots, s_{Nr_{d_1+\ldots+d_N}}^T(t)]^T \in \mathbb{R}^{nd},\end{aligned}$$

and

$$\begin{aligned}C =&[c_{1r_1}^T, c_{1r_2}^T, \ldots, c_{1r_{d_1}}^T, \ldots, c_{ir_{d_1+\ldots+d_{i-1}+1}}^T, \ldots,\\ &c_{ir_{d_1+\ldots+d_i}}^T, \ldots, c_{Nr_{d_1+\ldots+d_N}}^T]^T \in \mathbb{R}^{nd}.\end{aligned}$$

Without loss of generality, assume the subscript r_s in vector $\hat{x}(t)$ represents the neighbor j of agent i, i.e., $\hat{x}_{ir_s}(t) = \hat{x}_{ij}(t)$, where $r_s \in N_i$, $s \in \{d_1+d_2+\ldots+d_{i-1}+1, \ldots, d_1+\ldots+d_i\}$. Based on the above notations, we construct three matrices to establish the relation of the states of agents and their estimates.

$P_{m(t)}$ is designed to select each neighbor of each agent at time t. Define $P_{m(t)} = \mathrm{diag}\{p_{m(t)}^{11}, p_{m(t)}^{22}, \ldots, p_{m(t)}^{dd}\} \in \mathbb{R}^{d\times d}$, where $p_{m(t)}^{ss} = 1$ when $(i, r_s) \in E_{m(t)}$, else $p_{m(t)}^{ss} = 0$.

Q is designed to select the true state of the agent that correlates with its estimate. Define $Q = [Q_{1r_1}, \ldots, Q_{1r_{d_1}}, \ldots, Q_{Nr_{d_1+\ldots+d_{N-1}+1}}, \ldots, Q_{Nr_{d_1+\ldots+d_N}}]^T \in \mathbb{R}^{d\times N}$, where $Q_{ir_s} = Q_{ij} = [\vec{0}_{j-1}^T, 1, \vec{0}_{N-j}^T]^T \in \mathbb{R}^N$ for $(i, r_s) \in E$, else $Q_{ir_s} = \vec{0}_N$.

$W_{m(t)}$ is designed to select the neighbor set of each agent at time t. Define $W_{m(t)} = [W_{m(t)}^1, \ldots, W_{m(t)}^N]^T \in \mathbb{R}^{N\times d}$, where $W_{m(t)}^i = [\vec{0}_{d_1+\ldots+d_{i-1}}, b_1, \ldots, b_{d_i}, \vec{0}_{d_{i+1}+\ldots+d_N}]^T \in \mathbb{R}^d$ for $i \in \{1, \ldots, N\}$, $\forall k_i \in \{1, \ldots, d_i\}$, $b_{k_i} = 1$ when $(i, r_{k_i+d_1+\ldots+d_{i-1}}) \in E_{m(t)}$, else $b_{k_i} = 1$.

Based on the above matrices, the vector forms of estimation and update are given as follows:

1.Estimation:

$$\begin{aligned}\hat{x}(t) =&\boldsymbol{\Pi_M}\Big\{(I_d \otimes A)\hat{x}(t-1) + \frac{\beta}{t}(P_{m(t)} \otimes I_n)\big(\Phi_F(C-\\ &(I_d \otimes A)\hat{x}(t-1)) - s(t)\big)\Big\},\end{aligned} \tag{6}$$

where $\boldsymbol{\Pi_M}(z) = [\Pi_M^T(z_1), \ldots, \Pi_M^T(z_d)]^T$, $\Phi_F(z) = [\mathcal{F}^T(z_1), \ldots, \mathcal{F}^T(z_d)]^T$, for any $z = [z_1^T, z_2^T, \ldots, z_d^T]^T \in \mathbb{R}^{nd}$, $z_k \in \mathbb{R}^n$ for $k = 1, \ldots, d$.

2.Update:

$$\begin{aligned}x(t+1) =& \Big(I_N \otimes A - \frac{\gamma}{(t+1)d_{\max}} L_{m(t)} \otimes BB^T A\Big)x(t)\\ &+ \frac{\gamma}{(t+1)d_{\max}}(W_{m(t)} \otimes BB^T A)\varepsilon(t),\end{aligned} \tag{7}$$

where $\varepsilon(t) = \hat{x}(t) - (Q \otimes I_n)x(t)$ is the estimation error.

IV. Main Result

In this section, we will show that all agents can reach weak consensus and give the corresponding consensus rate.

To prove each agent can achieve weak consensus, we give the following lemmas first.

Lemma 1: ([26]). Denote $\check{L} = \sum_{i=1}^h p_i L_i$. If Assumption 1 holds, then matrix $\check{L}$ has the following properties:

i) $\check{L}$ is a nonnegative definite matrix with rank $n-1$.

ii) $\check{L}^2 \ge \frac{\lambda_2^2}{\lambda_N}\check{L}$, where λ_2 and λ_N are the smallest positive eigenvalue and the largest eigenvalue of $\check{L}$, respectively.

Lemma 2: The agent states $x_i(t)$ and the estimates $\hat{x}_{ij}(t)$ are all bounded, i.e., $\|x_i(t)\| \le M$ and $\|\hat{x}_{ij}(t)\| \le M$, where

M is the upper boundary for the norm of initial values, $i = 1, 2, \dots, N$, $j \in N_i^{m(t)}$, $t \geq t_0 + 1$.

Proof: First, due to the definition of M, we can get $\|x_i^0\| \leq M, \|x_{ij}^0\| \leq M$. By the estimation (3) and the definition of $\Pi_M(\cdot)$ (4), we have $\|\hat{x}_{ij}(t)\| \leq M$ for $t \geq t_0+1$.

Then, assume that $\|x_i(k)\| \leq M$ for $k = t_0+1, t_0+2, \dots, t$, we have

i) When there is no neighbor of the agent i at time t, by Remark 6, we can get $x_i(t+1) = Ax_i(t)$. Since A is an orthogonal matrix, we have $\|x_i(t+1)\| \leq \|A\|\|x_i(t)\| \leq \|x_i(t)\| \leq M$.

ii) When there exists neighbor of the agent i at time t, by Remark 6, we can get

$$\begin{aligned}
&\|x_i(t+1)\| \\
=&\Big\|Ax_i(t) + \frac{\gamma BB^T A}{(t+1)d_{\max}} \sum_{j\in N_i^{m(t)}} \big(\hat{x}_{ij}(t) - x_i(t)\big)\Big\| \\
=&\Big\|Ax_i(t) - \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}} Ax_i(t) + \frac{d_i^{m(t)}\gamma BB^T A}{(t+1)d_{\max}} \\
&\cdot \sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} \hat{x}_{ij}(t)\Big\| \\
=&\Big\|\big(I_n - \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}}\big)Ax_i(t) + \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}} \\
&\cdot A\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} \hat{x}_{ij}(t)\Big\|.
\end{aligned}$$

Since $d_i^{m(t)} > 0$ and $\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} = d_i^{m(t)} \frac{1}{d_i^{m(t)}} = 1$, we have $\big\|A\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} \hat{x}_{ij}(t)\big\| \leq \|A\| \big\|\sum_{j\in N_i^{m(t)}} \frac{1}{d_i^{m(t)}} \hat{x}_{ij}(t)\big\| \leq M$.

Moreover, for arbitrary γ, we can choose an initial time t_0 that satisfies $0 < \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}} < I_n$ when $t \geq t_0 + 1$. Since $\|(I_n - \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}}) + \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}}\| = \|I_n\| = 1$ and $\|Ax_i(t)\| \leq M$, we can get

$$\begin{aligned}
\|x_i(t+1)\| \leq& \Big\|\big(I_n - \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}}\big)M + \frac{d_i^{m(t)}\gamma BB^T}{(t+1)d_{\max}}M\Big\| \\
\leq& M.
\end{aligned}$$

Thus, by induction, we have $\|x_i(t)\| \leq M$ for all $t \geq t_0+1$. The lemma is proved. ∎

Then, to jointly analyze the structure of the topology graph and system model, we provide the following lemma.

Lemma 3: For positive semi-definite matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{m\times m} (i = 1, 2)$, if $A_1 \geq A_2$ and $B_1 \geq B_2$, then

$$A_1 \otimes B_1 \geq A_2 \otimes B_2.$$

Proof: For arbitrary positive semi-definite matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, denote $\lambda_1, \lambda_2, \dots, \lambda_n$ and $\mu_1, \mu_2, \dots, \mu_m$ as the eigenvalues of A and B, respectively. Then, by the Theorem 4.2.12 in [42], $\lambda_i \mu_j$ are the eigenvalues of $A \otimes B$, $i = 1, \dots, n$, $j = 1, \dots, m$. Since $A \geq 0$ and $B \geq 0$, we have $\lambda_i \geq 0$ and $\mu_j \geq 0$, thus $\lambda_i \mu_j \geq 0$.

Then, by the definition of the Kronecker product, we have $(A \otimes B)^T = A^T \otimes B^T = A \otimes B$, i.e., $A \otimes B$ is symmetric as well. This together with $\lambda_i \mu_j \geq 0$, yields $A \otimes B \geq 0$.

From the above conclusion and the distributive property of Kronecker product, we have

$$\begin{aligned}
A_1 \otimes B_1 - A_2 \otimes B_1 =&(A_1 - A_2) \otimes B_1 \geq 0, \\
A_2 \otimes B_1 - A_2 \otimes B_2 =&A_2 \otimes (B_1 - B_2) \geq 0.
\end{aligned}$$

Thus, we have $A_1 \otimes B_1 \geq A_2 \otimes B_1 \geq A_2 \otimes B_2$. ∎

Next, we introduce two Lyapunov functions, $V(t)$ and $R(t)$, to analyze the weak consensus of all agents and the convergence properties of estimates, respectively. These functions are defined as follows:

$$V(t) = E[x^T(t)(L_{m(t)} \otimes I_n)x(t)], \tag{8}$$

$$R(t) = E[\varepsilon^T(t)\varepsilon(t)]. \tag{9}$$

Then, the following two lemmas show the coupling relations of the two Lyapunov functions.

Lemma 4: Under Assumptions 1-4, $V(t)$ satisfies

$$\begin{aligned}
V(t) \leq \Big(1 - \frac{3\gamma\lambda_2^2\lambda_{ab}}{2t\lambda_N d_{\max}}\Big) V(t-1) + \frac{2\gamma\lambda_W\lambda_N\lambda_{AB}^2}{td_{\max}\lambda_2^2\lambda_{ab}} \\
\cdot R(t-1) + \frac{\hat{B}}{t^2},
\end{aligned}$$

where $\lambda_{ab} = \lambda_{\min}(A^TBB^TA)$, $\lambda_{AB} = \lambda_{\max}(A^TBB^TA)$, $\lambda_W = \max_{1\leq i\leq h} \{\lambda_{\max}\{W_i^T \check{L} W_i\}\}$, $\hat{B}$ is a constant.

Proof: See proof in Appendix I. ∎

Lemma 5: Under Assumptions 1-4, $R(t)$ satisfies

$$\begin{aligned}
R(t) \leq&\Big(1 - \frac{1}{td_{\max}}\big(2\beta p_{\min} f_M d_{\max} - \gamma\alpha\big)\Big)R(t-1) \\
&+ \frac{2\gamma\lambda_W\lambda_N\lambda_{AB}^2}{td_{\max}\lambda_2^2\lambda_{ab}} V(t-1) + \frac{\tilde{B}}{t^2},
\end{aligned}$$

where $p_{\min} = \min_{1\leq i\leq h}\{p_i\}$, $f_M = \min_{i,j,k} f(|c_{ijk}| + M)$, c_{ijk} is the kth element of c_{ij}, $\lambda_{QL} = \max_{1\leq i\leq h}\{\lambda_{\max}\{QL_iQ^T\}\}$, $\lambda_Q = \lambda_{\max}\{QQ^T\}$, $\lambda_{\check{W}} = \max_{1\leq i\leq h}\{\lambda_{\max}\{W_i^TW_i\}\}$, $\alpha = \frac{\lambda_2^2\lambda_{QL}\lambda_{ab}}{2\lambda_N\lambda_W} + 2\lambda_{AB}\sqrt{\lambda_Q\lambda_{\check{W}}}$, $\tilde{B}$ is a constant.

Proof: See proof in Appendix II. ∎

By Lemmas 4-5, we establish the relation between these two Lyapunov functions. Then, a new function $Z(t) = (V(t), R(t))^T$ is constructed to jointly analyze their properties, to overcome the difficulty resulting from the coupling of the estimation and control.

Lemma 6: ([26]). If Assumptions 1-4 hold, then

$$\|Z(t)\| \leq \|(I - \frac{1}{t}U)Z(t-1)\| + \frac{1}{t^2}\|H\|,$$

$$\|Z(t)\| = \begin{cases} O\left(\frac{1}{t^{\lambda_{\min}(U)}}\right), & \lambda_{\min}(U) < 1; \\ O\left(\frac{\ln t}{t}\right), & \lambda_{\min}(U) = 1; \\ O\left(\frac{1}{t}\right), & \lambda_{\min}(U) > 1, \end{cases}$$

where $U = \begin{bmatrix} u_1 & u_2 \\ u_2 & u_4 \end{bmatrix}$, $H = \big[\hat{B}, \tilde{B}\big]^T$, $u_1 = \frac{3\gamma\lambda_2^2\lambda_{ab}}{2\lambda_N d_{\max}}$, $u_2 = \frac{-2\gamma\lambda_W\lambda_N\lambda_{AB}^2}{d_{\max}\lambda_2^2\lambda_{ab}}$, $u_4 = \frac{1}{d_{\max}}\big(2\beta p_{\min} f_M d_{\max} - \gamma\alpha\big)$, α is the same as in Lemma 5.

Remark 7: Noticing that $0 \leq V(t) \leq \|Z(t)\|$ and $0 \leq R(t) \leq \|Z(t)\|$, we can transform the analysis of weak consensus and estimate convergence property into analyzing the convergence of $Z(t)$.

Theorem 1: Under Assumptions 1-4, the switching MAS (1)-(2) reach weak consensus and the estimates of states converge to the real states, i.e.,

$$\lim_{t\to\infty} E[\|x_i(t)-x_j(t)\|^2]=0,$$
$$\lim_{t\to\infty} E[\|\hat{x}_{ij}(t)-x_j(t)\|^2]=0,$$

when $\beta>\frac{1}{2p_{\min}f_M}\left(\frac{u_2^2}{u_1}+\frac{\gamma\alpha}{d_{\max}}\right)$ with u_1, u_2 and α being given in Lemmas 5-6.

Proof: Let $|\lambda I-U|=(\lambda-u_1)(\lambda-u_4)-u_2^2=0$. Then,

$$\lambda_{\min}(U)=\frac{1}{2}\Big(u_1+u_4-\sqrt{(u_1+u_4)^2-4(u_1u_4-u_2^2)}\Big).$$

If $\beta>\frac{1}{2p_{\min}f_M}\left(\frac{u_2^2}{u_1}+\frac{\gamma\alpha}{d_{\max}}\right)$, then we have $u_1u_4>u_2^2$. Since $u_1>0$ and $u_1u_4>u_2^2$, $\lambda_{\min}(U)>0$.

By Lemma 6, we have

$$\|Z(t)\|=\begin{cases}O\left(\frac{1}{t^{\lambda_{\min}(U)}}\right), & \lambda_{\min}(U)<1;\\ O\left(\frac{\ln t}{t}\right), & \lambda_{\min}(U)=1;\\ O\left(\frac{1}{t}\right), & \lambda_{\min}(U)>1,\end{cases}$$

And since $\lambda_{\min}(U)>0$, there is $\lim_{t\to\infty}\|Z(t)\|=0$.

By Remark 7, we have

$$\lim_{t\to\infty}V(t)=0,\ \lim_{t\to\infty}R(t)=0. \tag{10}$$

Denote the Laplacian matrix of G as L_G and $\sum_{i=1}^h L_i-L_G\triangleq L_{\sum-G}$. By the relation between $\{G_1,G_2,\dots,G_h\}$ and G, we know that $L_{\sum-G}$ is a Laplacian matrix of a weighted graph. Then, we have $\sum_{i=1}^h L_i-L_G=L_{\sum-G}\geq 0$, i.e., $\sum_{i=1}^h L_i\geq L_G$. Since $p_i>0$ and $\sum_{i=1}^h p_i=1$, we have

$$L_G\leq\sum_{i=1}^h L_i\leq\sum_{i=1}^h\frac{p_i}{p_{\min}}L_i.$$

By Assumption 1, $\{G_1,G_2,\dots,G_h\}$ are jointly connected, then there exists a road between any different agents i and j in the network G. Suppose the road is as follows:

$$i=r_0\to r_1\to r_2\to\cdots\to r_{p-1}\to r_p=j,\quad p\leq N$$

which implies $r_{i+1}\in N_{r_i}$. Then, the mean square error of any two different agents satisfies:

$$\begin{aligned}&E[\|x_i(t)-x_j(t)\|^2]\\ =&E[\|\big(x_{r_0}(t)-x_{r_1}(t)\big)+\big(x_{r_1}(t)-x_{r_2}(t)\big)\\ &+\cdots+\big(x_{r_{p-1}}(t)-x_{r_p}(t)\big)\|^2]\\ \leq& N\sum_{i=1}^N\sum_{j\in N_i}E[\|x_i(t)-x_j(t)\|^2]\\ \leq& 2NE[x^T(t)(L_G\otimes I_n)x(t)]\\ \leq& 2NE[x^T(t)(\sum_{i=1}^h\frac{p_i}{p_{\min}}L_i\otimes I_n)x(t)]=\frac{2N}{p_{\min}}V(t)\end{aligned} \tag{11}$$

Meanwhile,

$$\begin{aligned}&E[\|\hat{x}_{ij}(t)-x_j(t)\|^2]\\ \leq&\sum_{i=1}^N\sum_{j\in N_i}E[\|\hat{x}_{ij}(t)-x_j(t)\|^2]=R(t).\end{aligned} \tag{12}$$

Substituting (10) into (11)-(12) gives the theorem. ■

Theorem 2: Under Assumptions 1-4, the switching MAS (1)-(2) reach weak consensus at the rate of $O\left(\frac{1}{t}\right)$, and the covergence rate of the estimation error reach $O(\frac{1}{t})$, i.e.,

$$E[\|x_i(t)-x_j(t)\|^2]=O\left(\frac{1}{t}\right),$$
$$E[\|\hat{x}_{ij}(t)-x_j(t)\|^2]=O\left(\frac{1}{t}\right),$$

when $\beta>\frac{1}{2p_{\min}f_M}\left(\frac{u_2^2}{u_1-1}+\frac{\gamma\alpha}{d_{\max}}+1\right)$, $\gamma>\frac{2\lambda_N d_{\max}}{3\lambda_2^2\lambda_{ab}}$, with u_1, u_2 and α being given in Lemmas 5-6.

Proof: Similar to the proof of Theorem 1, if $\beta>\frac{1}{2p_{\min}f_M}\left(\frac{u_2^2}{u_1-1}+\frac{\gamma\alpha}{d_{\max}}+1\right)$, we have

$$u_4>\frac{u_2^2}{u_1-1}+1.$$

If $\gamma>\frac{2\lambda_N d_{\max}}{3\lambda_2^2\lambda_{ab}}$, then, $u_1-1>0$, $u_4(u_1-1)>u_2^2+u_1-1$. Therefore,

$$(u_1+u_4)^2-4(u_1u_4-u_2^2)<(u_1+u_4-2)^2,$$

and hence, we have $\lambda_{\min}(U)>1$. By Lemma 6, we have

$$\|Z(t)\|=O\left(\frac{1}{t}\right).$$

Then, by Remark 7, we have

$$V(t)=O\left(\frac{1}{t}\right),R(t)=O\left(\frac{1}{t}\right).$$

By (11)-(12), we can obtain the theorem. ■

Remark 8: Different from [23] and [26], this paper introduces the coefficient γ into the controller, which removes the previous constraint for the graph structure, such as $\frac{\lambda_2^2}{\lambda_N}>1$ in [23] and $\frac{3\lambda_2^2}{2\lambda_N d_{\max}}>1$ in [26]. This implies that, by selecting appropriate γ and β, the system can attain consensus and convergence rates of $O(\frac{1}{t})$, as long as the graphs are connected or jointly connected.

V. Numerical Simulation

This section will illustrate the theoretical results with a simulation example.

Consider a third-order MAS that has three agents, the state of the agent i is as follows:

$$x_i(t+1)=Ax_i(t)+Bu_i(t),\quad i=1,2,3,$$

where $A=\begin{bmatrix}0.5&0.5&-0.5\\-0.5&-0.5&-0.5\\1&1&0\end{bmatrix}$, $B=\begin{bmatrix}2&0&1\\0.5&1.5&0\\0&0&2\end{bmatrix}$. A is not orthogonal but neutrally stable, as indicated in Remark 3, since its eigenvalues are $\{0,i,-i\}$. On the other hand, B is of full row rank, satisfying Assumption 4. Therefore, both A and B are satisfying the underlying conditions of this paper.

The switching topologies are shown in Figure 1, with $p_1=7/24$, $p_2=1/3$, $p_3=3/8$. These topologies are jointly connected and satisfy Assumption 1.

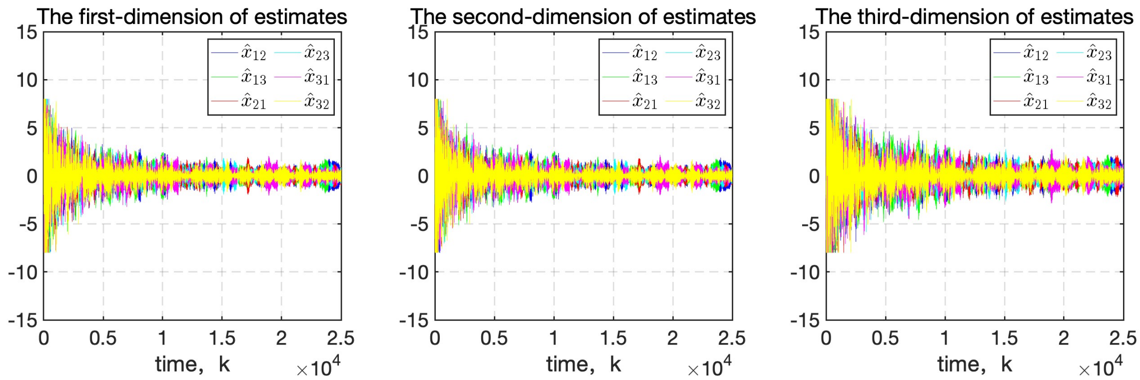

Fig. 2. The estimates of neighbors' states

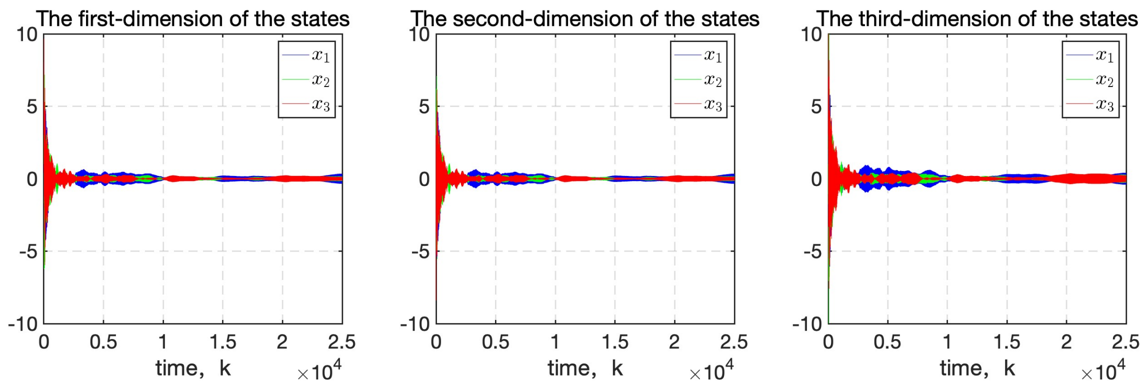

Fig. 3. The states of agents

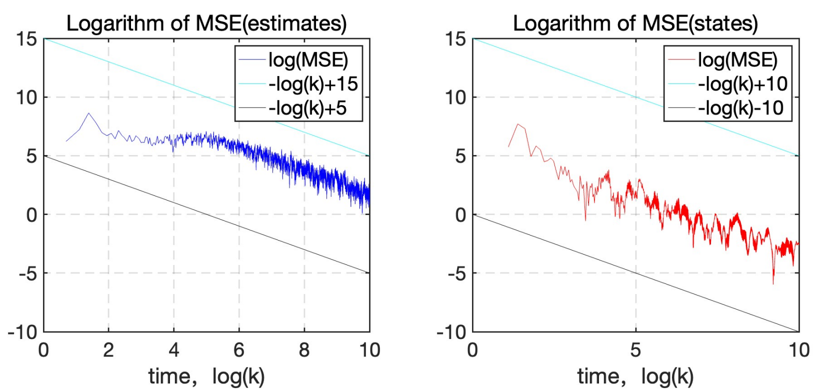

Fig. 4. The trajectory of the logarithm of MSE

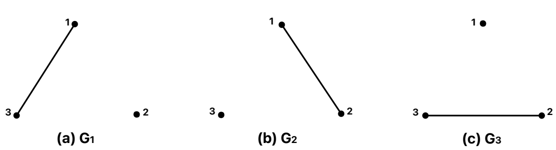

Fig. 1. switching topologies

Besides, we assume the communication noises between agents are distributed as $N(0, 64 \cdot I_3)$, which satisfies Assumptions 2-3. Take the initial state as $x^0 = (2, 2, 2, 1, 1, 1, 3, 3, 3)^T$, and the initial estimate as $\hat{x}^0 = (0, 0, 0, 2, 2, 2, 2, 2, 2, -3, -3, -3, 1, 1, 1, -2, -2, -2)^T$, and set the boundary $M = 8$. Then, by Theorems 1-2, set $\beta = 3000$ and $\gamma = 0.9$, using the Estimation-Based Consensus Algorithm, one can get the following simulation results.

As shown in Figs. 2-3, the states of all agents reach consensus, and the estimates of the neighbors' states also approach their real states. Besides, Fig. 4 illustrates that the estimation errors can converge to 0 at the rate of $O(\frac{1}{t})$ and each agent can reach weak consensus at the same rate.

VI. Conclusions

The consensus problem of high-order MASs with binary-valued communications and switching topologies is studied in this paper. An estimation-based consensus algorithm, consisting of estimation and control, is designed. And, a method that jointly analyzes the structure of the topology graph and system model is employed to overcome the complexity of high-order MASs. By constructing and analyzing two Lyapunov functions about estimation error and consensus error, this paper overcomes the difficulty resulting from the coupling of the estimation and control and proves that the estimation error can converge to zero and all agents can reach weak consensus. Moreover, it is also shown that the rate of convergence and consensus can both reach the reciprocal of the iteration times.

In the future, there will be many interesting problems in the consensus problem of high-order MAS under binary-valued communications. For example, if the coefficient matrix A is unstable, is the algorithm described in this paper still valid? If not, how can we improve the algorithm?

Appendix I
The proof of Lemma 4

Let

$$\begin{aligned}V_1 =&E\Big[x^T(t-1)\Big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes A^TBB^T\Big)\big(L_{m(t)}\\&\otimes I_n\big)\Big(I_N\otimes A-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes BB^TA\Big)x(t-1)\Big],\\V_2 =&\frac{2\gamma}{td_{max}}E\Big[x^T(t-1)\Big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes A^TBB^T\Big)\\&\cdot\big(L_{m(t)}\otimes I_n\big)\big(W_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\Big],\\V_3 =&\frac{\gamma^2}{t^2d^2_{max}}E\Big[\varepsilon^T(t-1)\big(W^T_{m(t-1)}\otimes A^TBB^T\big)\big(L_{m(t)}\otimes I_n\big)\\&\cdot\big(W_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\Big].\end{aligned}$$

Then, from (7)-(8), it follows that

$$V(t)=E[x^T(t)(L_{m(t)}\otimes I_n)x(t)]=V_1+V_2+V_3. \quad \text{(A1)}$$

Firstly, by the property of conditional expectation, we get $E[x^T(t)(L_{m(t)}\otimes I_n)x(t)] = E\big[E[x^T(t)(L_{m(t)}\otimes I_n)x(t)\big|x(t)]\big] = E[x^T(t)(\check{L}\otimes I_n)x(t)]$. Similarly, by Assumptions 2-3, we have

$$\begin{aligned}V_1 =&E\Big[E\big[x^T(t-1)\big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes A^TBB^T\big)\\&\cdot\big(\check{L}\otimes I_n\big)\big(I_N\otimes A-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes BB^TA\big)x(t-1)\\&\big|x(t-1)\big]\Big]\\=&E\Big[x^T(t-1)\Big(\check{L}\otimes I_n-\frac{2\gamma\check{L}^2\otimes A^TBB^TA}{td_{\max}}+\frac{\gamma^2}{t^2d^2_{\max}}\\&\cdot E\big[(L_{m(t-1)}\check{L}L_{m(t-1)})\otimes A^TBB^TBB^TA\big]\Big)x(t-1)\Big].\end{aligned}$$

Moreover, by Lemmas 1-3, we have $\check{L}^2\otimes A^TBB^TA\geq(\frac{\lambda_2^2}{\lambda_N}\check{L})\otimes(\lambda_{ab}I_n)=\frac{\lambda_2^2\lambda_{ab}}{\lambda_N}\check{L}\otimes I_n$, and

$$\begin{aligned}&E\big[x^T(t-1)\big(\check{L}\otimes I_n-\frac{2\gamma\check{L}^2\otimes A^TBB^TA}{td_{\max}}\big)x(t-1)\big]\\\leq&(1-\frac{2\gamma\lambda_2^2\lambda_{ab}}{t\lambda_Nd_{\max}})E[x^T(t-1)(\check{L}\otimes I_n)x(t-1)].\end{aligned}$$

Then, by Lemma 2, we have

$$V_1\leq(1-\frac{2\gamma\lambda_2^2\lambda_{ab}}{t\lambda_Nd_{\max}})V(t-1)+\frac{B_1}{t^2}, \quad \text{(A2)}$$

where $0<B_1<\infty$.

Similarly, by Assumptions 2-3, we obtain

$$\begin{aligned}V_2 =&\frac{2\gamma}{td_{max}}E\Big[E\big[x^T(t-1)\big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{max}}\otimes A^TB\\&\cdot B^T\big)\big(L_{m(t)}\otimes I_n\big)\big(W_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\\&\big|x(t-1),\hat{x}(t-1),L_{m(t-1)}\big]\Big]\\=&\frac{2\gamma}{td_{\max}}E\Big[x^T(t-1)\Big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{\max}}\otimes A^TBB^T\Big)\\&\cdot\big(\check{L}\otimes I_n)(W_{m(t-1)}\otimes BB^TA)\varepsilon(t-1)\Big].\end{aligned}$$

Since $\check{L}$ is a positive semi-definite matrix, there exists a matrix $\tilde{L}$ such that $\check{L}=\tilde{L}^T\tilde{L}$. Then, substituting this decomposition into the above equation and using the Schwarz inequality gives

$$\begin{aligned}V_2\leq&\frac{2\gamma}{td_{\max}}\Big(E\Big[x^T(t-1)\big(I_N\otimes A^T-\frac{\gamma L_{m(t-1)}}{td_{\max}}\otimes A^TBB^T\big)\\&\cdot\big(\tilde{L}^T\otimes I_n\big)\big(\tilde{L}\otimes I_n\big)\big(I_N\otimes A-\frac{\gamma L_{m(t-1)}}{td_{\max}}\otimes BB^TA\big)\\&x(t-1)\Big]\Big)^{\frac{1}{2}}\cdot\Big(E\Big[\varepsilon^T(t-1)\big(W^T_{m(t-1)}\otimes A^TBB^T\big)\\&\big(\tilde{L}^T\tilde{L}\otimes I_n\big)\cdot\big(W_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\Big]\Big)^{\frac{1}{2}}.\end{aligned}$$

By Assumption 4, we have $AA^T=I_n$, $\lambda(B)\neq0$, and then, $\lambda(A^TBB^TBB^TA)=\lambda(A^TBB^TAA^TBB^TA)=\lambda^2(A^TBB^TA)$. By $\lambda(W^T_{m(t-1)}\check{L}W_{m(t-1)})\leq\lambda_W$, $\lambda(A^TBB^TBB^TA)\leq\lambda^2_{AB}$, and Lemma 3, we have

$$\begin{aligned}V_2\leq&\frac{2\gamma}{td_{\max}}\sqrt{\Big[\big(1-\frac{2\gamma\lambda_2^2\lambda_{ab}}{t\lambda_Nd_{\max}}\big)V(t-1)\Big]\cdot[\lambda_W\lambda^2_{AB}R(t-1)]}\\&+\frac{B_2}{t^2}\\\leq&\frac{2\gamma}{td_{\max}}\sqrt{V(t-1)\lambda_W\lambda^2_{AB}R(t-1)}+\frac{B_2}{t^2}\\\leq&\frac{\gamma}{td_{\max}}\Big(\frac{\lambda_2^2\lambda_{ab}}{2\lambda_N}V(t-1)+\frac{2\lambda_W\lambda^2_{AB}\lambda_N}{\lambda_2^2\lambda_{ab}}R(t-1)\Big)\\&+\frac{B_2}{t^2}, \quad \text{(A3)}\end{aligned}$$

when $t>\max\{\frac{2\gamma\lambda_2^2\lambda_{ab}}{\lambda_Nd_{\max}},t_0\}$, where $0<B_2<\infty$.

Since Q is a constant matrix, $\varepsilon(t)$ is bounded. Moreover, A is a constant matrix and $W_{m(t)}$ is finite, thus

$$V_3\leq\frac{B_3}{t^2}, \quad \text{(A4)}$$

where $0<B_3<\infty$.

By (A1)-(A4), we can obtain the lemma.

Appendix II
The proof of Lemma 5

Let

$$R_2 = \frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)(P_{m(t)}\otimes I_n) \cdot \big(\Phi_F(C-(I_d\otimes A)\hat{x}(t-1))-s(t)\big)\Big],$$

$$R_3 = \frac{2\gamma}{td_{\max}}E\Big[\varepsilon^T(t-1)\big((QL_{m(t-1)}\otimes A^TBB^TA) \cdot x(t-1)-(QW_{m(t-1)}\otimes A^TBB^TA)\varepsilon(t-1)\big)\Big].$$

Then from (6), (9), Remark 5 and the definition of $\varepsilon(t)$, we have

$$\begin{aligned} R(t) &= E[\varepsilon^T(t)\varepsilon(t)] \\ &\le E\Big[\Big((I_d\otimes A)\hat{x}(t-1)+\frac{\beta}{t}(P_{m(t)}\otimes I_n)\big(\Phi_F(C-(I_d\otimes A) \cdot \hat{x}(t-1))-s(t)\big)-(Q\otimes I_n)x(t)\Big)^T \\ &\quad \cdot\Big((I_d\otimes A)\hat{x}(t-1)+\frac{\beta}{t}(P_{m(t)}\otimes I_n)\big(\Phi_F(C-(I_d\otimes A) \cdot \hat{x}(t-1))-s(t)\big)-(Q\otimes I_n)x(t)\Big)\Big] \\ &= E\Big[\Big(\varepsilon^T(t-1)(I_d\otimes A^T)+\frac{\beta}{t}\big((P_{m(t)}\otimes I_n)\big(\Phi_F(C-(I_d\otimes A)\hat{x}(t-1))-s(t)\big)\big)^T+\frac{\gamma}{td_{max}}\big[x^T(t-1)(L_{m(t-1)}Q^T\otimes A^TBB^T)-\varepsilon^T(t-1)(W^T_{m(t-1)}Q^T\otimes A^TBB^T)\big]\Big) \\ &\quad \cdot\Big((I_d\otimes A)\varepsilon(t-1)+\frac{\beta}{t}(P_{m(t)}\otimes I_n)\big(\Phi_F(C-(I_d\otimes A) \cdot \hat{x}(t-1))-s(t)\big)+\frac{\gamma}{td_{max}}\big[(QL_{m(t-1)}\otimes BB^TA) \cdot x(t-1)-(QW_{m(t-1)}\otimes BB^TA)\varepsilon(t-1)\big]\Big)\Big] \\ &\le R(t-1)+R_2+R_3+\frac{B_4}{t^2}, \end{aligned} \tag{B1}$$

where $0<B_4<\infty$.

By the definition of $s(t)$, we can obtain that $E[s(t)]=\Phi_F\big(C-(Q\otimes I_n)x(t)\big)$. Then, using the property of conditional expectation, we get

$$\begin{aligned} R_2 &= \frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big(P_{m(t)}\otimes I_n\big)\Big(\Phi_F\big(C-(I_d\otimes A)\hat{x}(t-1)\big)-\Phi_F\big(C-(Q\otimes I_n)x(t)\big)\Big)\Big] \\ &= \frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big((\sum_{i=1}^h p_iP_i)\otimes I_n\big)\Big(\Phi_F\big(C-(I_d\otimes A)\hat{x}(t-1)\big)-\Phi_F\big(C-(Q\otimes I_n)x(t)\big)\Big)\Big]. \end{aligned}$$

And, denoting $\vec{f}=\frac{d\mathcal{F}}{dx}$, by Lagrange's Mean Value Theorem, we have

$$\begin{aligned} &\mathcal{F}\big(c_{ij}-A\hat{x}_{ij}(t-1)\big)-\mathcal{F}\big(c_{ij}-x_j(t)\big) \\ &=-\vec{f}(\xi_{ij}(t))\big(A\hat{x}_{ij}(t-1)-x_j(t)\big), \end{aligned}$$

where $\xi_{ij}(t)$ is between $c_{ij}-A\hat{x}_{ij}(t-1)$ and $c_{ij}-x_j(t)$.

Then, let $\xi(t)=(\xi^T_{1r_1}(t),\dots,\xi^T_{ir_s}(t),\dots,\xi^T_{Nr_{d_1+\dots+d_N}}(t))^T$, with r_s representing the neighbor j of agent i, i.e., $\xi_{ir_s}(t)=\xi_{ij}(t)$. Denote $\operatorname{diag}(\Phi_f(\xi(t)))$ as a diagonal matrix generated by each dimension of $\Phi_f(\xi(t))$, with $\Phi_f=\frac{d\Phi_F}{dx}$. By Lemma 2, $\xi_{ij}(t)$ is bounded. Since the function Φ_f is continuous, we have $\operatorname{diag}\big(\Phi_f(\xi(t))\big)\ge f_M\cdot I_{nd}$ and

$$\begin{aligned} &\Phi_F\big(C-(I_d\otimes A)\hat{x}(t-1)\big)-\Phi_F\big(C-(Q\otimes I_n)x(t)\big) \\ &=-\operatorname{diag}\big(\Phi_f(\xi(t))\big)\big((I_d\otimes A)\hat{x}(t-1)-(Q\otimes I_n)x(t)\big) \\ &=-\operatorname{diag}\big(\Phi_f(\xi(t))\big)\Big((I_d\otimes A)\varepsilon(t-1)+\frac{\gamma}{td_{\max}}\Big[\big(QL_{m(t-1)}\otimes BB^TA\big)x(t-1)-\big(QW_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\Big]\Big). \end{aligned}$$

Then, we have

$$\begin{aligned} R_2 &= -\frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big((\sum_{i=1}^h p_iP_i)\otimes I_n\big) \cdot \operatorname{diag}\big(\Phi_f(\xi(t))\big)\Big((I_d\otimes A)\varepsilon(t-1)+\frac{\gamma}{td_{\max}}\Big[\big(QL_{m(t-1)}\otimes BB^TA\big)x(t-1)-\big(QW_{m(t-1)}\otimes BB^TA\big)\varepsilon(t-1)\Big]\Big)\Big] \\ &= -\frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big((\sum_{i=1}^h p_iP_i)\otimes I_n\big) \cdot \operatorname{diag}\big(\Phi_f(\xi(t))\big)(I_d\otimes A)\varepsilon(t-1)\Big]+\frac{B_5}{t^2}, \end{aligned}$$

where $0<B_5<\infty$.

Subsequently, by Assumption 1 and the definition of $P_{m(t)}$, we can get $\sum_{i=1}^h P_i\ge I_d$, and then

$$\begin{aligned} R_2 &= -\frac{2\beta}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big((\sum_{i=1}^h p_iP_i)\otimes I_n\big) \cdot \operatorname{diag}\big(\Phi_f(\xi(t))\big)(I_d\otimes A)\varepsilon(t-1)\Big]+\frac{B_5}{t^2} \\ &\le -\frac{2\beta f_M}{t}E\Big[\varepsilon^T(t-1)(I_d\otimes A^T)\big((\sum_{i=1}^h p_iP_i)\otimes I_n\big) \cdot (I_d\otimes A)\varepsilon(t-1)\Big]+\frac{B_5}{t^2} \\ &\le -\frac{2\beta p_{\min}f_M}{t}R(t-1)+\frac{B_5}{t^2}. \end{aligned} \tag{B2}$$

In a similar way to V_2, let $L_{m(t-1)}=\tilde{L}^T_{m(t-1)}\tilde{L}_{m(t-1)}$ and use the Schwarz inequality again. Then, we have

$$\begin{aligned} &\frac{2\gamma}{td_{\max}}E[\varepsilon^T(t-1)(QL_{m(t-1)}\otimes A^TBB^TA)x(t-1)] \\ &= E[\varepsilon^T(t-1)(Q\tilde{L}^T_{m(t-1)}\otimes A^TBB^T)(\tilde{L}_{m(t-1)}\otimes A)x(t-1)] \\ &\le \frac{2\gamma}{td_{\max}}\Big(E[\varepsilon^T(t-1)\big((Q\tilde{L}^T_{m(t-1)}\tilde{L}_{m(t-1)}Q^T)\otimes A^TBB^T \cdot BB^TA\big)\varepsilon(t-1)]E[x^T(t-1)(\tilde{L}^T_{m(t-1)}\tilde{L}_{m(t-1)}\otimes I_n) \cdot x(t-1)]\Big)^{1/2} \\ &\le \frac{2\gamma}{td_{\max}}\sqrt{\lambda_{QL}\lambda^2_{AB}R(t-1)V(t-1)} \\ &\le \frac{\gamma}{td_{\max}}\Big(\frac{\lambda_{QL}\lambda_2^2\lambda_{ab}}{2\lambda_N\lambda_W}R(t-1)+\frac{2\lambda_N\lambda_W\lambda^2_{AB}}{\lambda_2^2\lambda_{ab}}V(t-1)\Big). \end{aligned}$$

Since $\lambda_{\tilde{W}}$ and λ_Q are finite, we have

$$-\frac{2\gamma}{td_{\max}}E[\varepsilon^T(t-1)(QW_{m(t-1)}\otimes A^TBB^TA)\varepsilon(t-1)]$$

$$
\begin{aligned}
\leq&\frac{2\gamma}{td_{max}}\Big(E[\varepsilon^T(t-1)(QQ^T\otimes A^TBB^TBB^TA)\varepsilon(t-1)]\Big)^{\frac{1}{2}}\\
&\cdot\Big(E[\varepsilon^T(t-1)(W^T_{m(t-1)}W_{m(t-1)}\otimes I_n)\varepsilon(t-1)]\Big)^{\frac{1}{2}}\\
\leq&\frac{2\gamma\lambda_{AB}\sqrt{\lambda_Q\lambda_{\breve{W}}}}{td_{\max}}R(t-1).
\end{aligned}
$$

Then, we can obtain

$$
\begin{aligned}
R_3\leq&\frac{\gamma}{td_{\max}}\Big(\frac{\lambda_{QL}\lambda_2^2\lambda_{ab}}{2\lambda_N\lambda_W}R(t-1)+\frac{2\lambda_N\lambda_W\lambda_{AB}^2}{\lambda_2^2\lambda_{ab}}V(t-1)\Big)\\
&+\frac{2\gamma\lambda_{AB}\sqrt{\lambda_Q\lambda_{\breve{W}}}}{td_{\max}}R(t-1)\\
=&\frac{\gamma\alpha}{td_{\max}}R(t-1)+\frac{2\gamma\lambda_N\lambda_W\lambda_{AB}^2}{td_{\max}\lambda_2^2\lambda_{ab}}V(t-1).
\end{aligned}
$$

This together with (B1)-(B2) gives the lemma.

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